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# SUPER CONGRUENCES AND ELLIPTIC CURVES OVER $\mathbb{F}_p$

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ABSTRACT. In this paper we deduce some new super-congruences motivated by elliptic curves over  $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ , where  $p > 3$  is a prime. Let  $d \in \{0, 1, \dots, (p-1)/2\}$ . We show that

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k} \binom{2k}{k+d}}{8^k} \equiv 0 \pmod{p} \quad \text{whenever } d \equiv \frac{p+1}{2} \pmod{2},$$

and

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k} \binom{2k}{k+d}}{16^k} \equiv \left(\frac{-1}{p}\right) + p^2 \frac{(-1)^d}{4} E_{p-3} \left(d + \frac{1}{2}\right) \pmod{p^3},$$

where  $E_{p-3}(x)$  denotes the Euler polynomial of degree  $p-3$ , and  $(-)$  stands for the Legendre symbol. The paper also contains some other results such as

$$\sum_{k=0}^{p-1} k^{(1+(\frac{-1}{p}))/2} \frac{\binom{6k}{3k} \binom{3k}{k}}{864^k} \equiv 0 \pmod{p^2}.$$

## 1. INTRODUCTION

Let  $p > 3$  be a prime and let  $\lambda$  be a rational  $p$ -adic integer (whose denominator is not divisible by  $p$ ). Consider the cubic curve in the Legendre form

$$\mathbb{E}_p(\lambda) : y^2 = x(x-1)(x-\bar{\lambda})$$

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over the finite field  $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ , where  $\bar{\lambda}$  is the residue class of  $\lambda \bmod p$ . This is an elliptic curve if  $\lambda \not\equiv 0, 1 \pmod{p}$ . Clearly the number of points on  $\mathbb{E}_p(\lambda)$  (including the point at infinity) is

$$\begin{aligned} \#\mathbb{E}_p(\lambda) &= 1 + \#\{(x, y) : 0 \leq x, y < p \text{ and } y^2 \equiv x(x-1)(x-\lambda) \pmod{p}\} \\ &= 1 + \sum_{x=0}^{p-1} \left( 1 + \left( \frac{x(x-1)(x-\lambda)}{p} \right) \right) = p + 1 + a_p(\lambda), \end{aligned}$$

where  $(\frac{\cdot}{p})$  denotes the Legendre symbol and

$$a_p(\lambda) := \sum_{x=0}^{p-1} \left( \frac{x(x-1)(x-\lambda)}{p} \right).$$

In this paper we propose the study of the weighted number  $N_p^{(d)}(\lambda)$  of points  $(x, y)$  in  $\mathbb{E}_p(\lambda)$  with weight  $x^d$  where  $d \in \mathbb{N} = \{0, 1, 2, \dots\}$  and  $d \leq (p-1)/2$ . For  $d = 1, \dots, (p-1)/2$ , Clearly

$$N_p^{(d)}(\lambda) = 1 + \sum_{x=0}^{p-1} x^d \left( 1 + \left( \frac{x(x-1)(x-\lambda)}{p} \right) \right) \equiv 1 + a_p^{(d)}(\lambda) \pmod{p}$$

where

$$a_p^{(d)}(\lambda) := \sum_{x=0}^{p-1} x^d \left( \frac{x(x-1)(x-\lambda)}{p} \right). \quad (1.1)$$

Concerning  $a_p^{(d)}(\lambda) \bmod p$  we have the following result.

**Theorem 1.1.** *Let  $p$  be an odd prime and let  $d \in \{0, \dots, (p-1)/2\}$ . Then, for any rational  $p$ -adic integer  $\lambda$  we have*

$$a_p^{(d)}(\lambda) \equiv (-1)^{(p+1)/2} \frac{\lambda^d}{4^d} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k} \binom{2(k+d)}{k+d}}{16^k} \lambda^k - \delta_{d, (p-1)/2} \pmod{p}. \quad (1.2)$$

Let  $p$  be an odd prime and let  $d \in \{0, \dots, (p-1)/2\}$ . Clearly

$$a_p^{(d)}(1) = \sum_{x=0}^{p-1} x^d \left( \frac{x}{p} \right) - 1 \equiv \sum_{x=1}^{p-1} x^{d+(p-1)/2} - 1 \equiv -\delta_{d, (p-1)/2} - 1 \pmod{p}.$$

Thus (1.2) with  $\lambda = 1$  gives the congruence

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k} \binom{2k+2d}{k+d}}{16^k} \equiv 4^d \left( \frac{-1}{p} \right) \pmod{p}.$$

However, we find that this congruence even holds modulo  $p^2$ .

**Theorem 1.2.** *Let  $p > 3$  be a prime and let  $d \in \{0, \dots, (p-1)/2\}$ . Then*

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k} \binom{2k+2d}{k+d}}{16^k} \equiv 4^d \left( \frac{-1}{p} \right) \pmod{p^2}, \quad (1.3)$$

and moreover

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k} \binom{2k}{k+d}}{16^k} \equiv \left( \frac{-1}{p} \right) + p^2 \frac{(-1)^d}{4} E_{p-3} \left( d + \frac{1}{2} \right) \pmod{p^3}, \quad (1.4)$$

where  $E_{p-3}(x)$  denotes the Euler polynomial of degree  $p-3$ .

(1.3) in the case  $d = 0$  was first conjectured by Rodriguez-Villegas [RV] in 2003 and later proved by Mortenson [M1] via an advanced tool involving the  $p$ -adic Gamma function and the Gross-Koblitz formula for character sums. (See also S. Ahlgren [A] and K. Ono [O] for such an approach.) (1.4) with  $d = 0$  yields the congruence

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{16^k} \equiv \left( \frac{-1}{p} \right) + p^2 E_{p-3} \pmod{p^3}$$

which was first proved in [S4] with the help of the software **Sigma**.

Let  $p \equiv 1 \pmod{4}$  be a prime. It is well known that  $p = x^2 + y^2$  for some  $x, y \in \mathbb{Z}$  with  $x \equiv 1 \pmod{4}$  and  $y \equiv 0 \pmod{2}$ . A celebrated result of Gauss asserts that  $\binom{(p-1)/2}{(p-1)/4} \equiv 2x \pmod{p}$ . This was refined in [CDP] as follows:

$$\binom{(p-1)/2}{(p-1)/4} \equiv \frac{2^{p-1} + 1}{2} \left( 2x - \frac{p}{2x} \right) \pmod{p^2}.$$

Recently the author's twin brother Z. H. Sun [Su] confirmed the author's following conjecture (cf. [S3, Conjecture 5.5]):

$$\begin{aligned} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{8^k} &\equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{(-16)^k} \equiv \left( \frac{2}{p} \right) \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{32^k} \\ &\equiv \left( \frac{2}{p} \right) \left( 2x - \frac{p}{2x} \right) \pmod{p^2}. \end{aligned}$$

In [S5] the author showed that

$$\begin{aligned} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k} C_k}{8^k} &\equiv -2 \sum_{k=0}^{p-1} \frac{k \binom{2k}{k}^2}{8^k} \\ &\equiv \frac{1}{2} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k} C_k}{(-16)^k} \equiv -4 \sum_{k=0}^{(p-1)/2} \frac{k \binom{2k}{k}^2}{(-16)^k} \equiv \left( \frac{2}{p} \right) \left( 2x - \frac{p}{x} \right) \pmod{p^2}, \end{aligned}$$

where  $C_k$  denotes the Catalan number  $\binom{2k}{k}/(k+1) = \binom{2k}{k} - \binom{2k}{k+1}$ . (Note that Catalan numbers occur naturally in many enumeration problems in combinatorics, see, e.g., [St, pp. 219–229].)

Motivated by (1.2) in the cases  $\lambda = -1, 2$  we obtain the following result.

**Theorem 1.3.** *Let  $p$  be an odd prime.*

(i) *If  $p \equiv 3 \pmod{4}$ , then*

$$\begin{aligned} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k} C_k}{8^k} &\equiv -2 \sum_{k=0}^{(p-1)/2} \frac{k \binom{2k}{k}^2}{8^k} \\ &\equiv -\frac{1}{2} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k} C_k}{(-16)^k} \equiv 4 \sum_{k=0}^{(p-1)/2} \frac{k \binom{2k}{k}^2}{(-16)^k} \\ &\equiv \frac{(-1)^{(p+1)/4}}{2} \binom{(p+1)/2}{(p+1)/4} \pmod{p} \end{aligned} \quad (1.5)$$

and

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{8^k} \equiv - \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{(-16)^k} \equiv \frac{2p(-1)^{(p+1)/4}}{\binom{(p+1)/2}{(p+1)/4}} \pmod{p^2}. \quad (1.6)$$

(ii) *We have*

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k} \binom{2k}{k+d}}{8^k} \equiv 0 \pmod{p} \quad (1.7)$$

for all  $d \in \{0, \dots, (p-1)/2\}$  with  $d \equiv (p+1)/2 \pmod{2}$ .

Besides (1.3) with  $d = 0$ , Rodriguez-Villegas [RV] also raised the following similar conjectures (confirmed in [M1–M3]) on super congruences with  $p$  a prime greater than 3:

$$\sum_{k=0}^{p-1} \frac{\binom{3k}{k} \binom{2k}{k}}{27^k} \equiv \left(\frac{p}{3}\right) \pmod{p^2}, \quad (1.8)$$

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{2k} \binom{2k}{k}}{64^k} \equiv \left(\frac{-2}{p}\right) \pmod{p^2}, \quad (1.9)$$

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k} \equiv \left(\frac{-1}{p}\right) \pmod{p^2}. \quad (1.10)$$

Note that the denominators 27, 64, 432 come from the following observation via the Stirling formula:

$$\binom{3k}{k} \binom{2k}{k} \sim \frac{\sqrt{3} 27^k}{2k\pi}, \quad \binom{4k}{2k} \binom{2k}{k} \sim \frac{64^k}{\sqrt{2}k\pi}, \quad \binom{6k}{3k} \binom{3k}{k} \sim \frac{432^k}{2k\pi}.$$

Up to now no simple proofs of (1.8)-(1.10) have been found.

Motivated by the work in [PS] and [ST], the author [S2] determined  $\sum_{k=0}^{p-1} \binom{2k}{k}/m^k$  modulo  $p^2$  in terms of Lucas sequences, where  $p$  is an odd prime and  $m$  is an integer not divisible by  $p$ . In [S3] and [S4] the author raised many conjectures on sums of terms involving central binomial coefficients.

Let  $\mathbb{N} = \{0, 1, 2, \dots\}$ . For a sequence of  $(a_n)_{n \in \mathbb{N}}$  of numbers, as in [S1] we introduce its dual sequence  $(a_n^*)_{n \in \mathbb{N}}$  by defining

$$a_n^* := \sum_{k=0}^n \binom{n}{k} (-1)^k a_k \quad (n = 0, 1, 2, \dots).$$

It is well-known that  $(a_n^*)^* = a_n$  for all  $n \in \mathbb{N}$  (see, e.g., (5.48) of [GKP, p. 192]). For Bernoulli numbers  $B_0, B_1, B_2, \dots$ , the sequence  $((-1)^n B_n)_{n \in \mathbb{N}}$  is self-dual.

**Theorem 1.4.** *Let  $p > 3$  be a prime and let  $(a_n)_{n \in \mathbb{N}}$  be any sequence of  $p$ -adic integers. Then we have*

$$\sum_{k=0}^{p-1} \frac{\binom{3k}{k} \binom{2k}{k}}{27^k} a_k \equiv \left(\frac{p}{3}\right) \sum_{k=0}^{p-1} \frac{\binom{3k}{k} \binom{2k}{k}}{27^k} a_k^* \pmod{p^2}, \quad (1.11)$$

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{2k} \binom{2k}{k}}{64^k} a_k \equiv \left(\frac{-2}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{4k}{2k} \binom{2k}{k}}{64^k} a_k^* \pmod{p^2}, \quad (1.12)$$

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k} a_k \equiv \left(\frac{-1}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k} a_k^* \pmod{p^2}. \quad (1.13)$$

*Remark.* Z. H. Sun [Su] recently proved that

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{16^k} \left( a_k - \left(\frac{-1}{p}\right) a_k^* \right) \equiv 0 \pmod{p^2}$$

for any odd prime  $p$  via Legendre polynomials. We can also show for any prime  $p > 3$  the following result similar to (1.3) and (1.4): If  $d \in \{0, \dots, \lfloor p/3 \rfloor\}$  then

$$\frac{1}{4^d} \sum_{k=0}^{(p-1)/2} \frac{\binom{3k}{k} \binom{2k+2d}{k+d}}{27^k} \equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{3k}{k} \binom{2k}{k+d}}{27^k} \equiv \left(\frac{p}{3}\right) \pmod{p};$$

if  $d \in \{0, \dots, \lfloor p/4 \rfloor\}$  then

$$\frac{1}{4^d} \sum_{k=0}^{(p-1)/2} \frac{\binom{4k}{2k} \binom{2k+2d}{k+d}}{64^k} \equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{4k}{2k} \binom{2k}{k+d}}{64^k} \equiv \left(\frac{-2}{p}\right) \pmod{p}.$$

Since  $(1-x)^k = \sum_{j=0}^k \binom{k}{j} (-1)^j x^j$ , by Theorem 1.4 we have the following result.

**Theorem 1.5.** *Let  $p > 3$  be a prime. Then, in the ring  $\mathbb{Z}_p[x]$  we have*

$$\sum_{k=0}^{p-1} \frac{\binom{3k}{k} \binom{2k}{k}}{27^k} \left( x^k - \left( \frac{p}{3} \right) (1-x)^k \right) \equiv 0 \pmod{p^2}, \quad (1.14)$$

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{2k} \binom{2k}{k}}{64^k} \left( x^k - \left( \frac{-2}{p} \right) (1-x)^k \right) \equiv 0 \pmod{p^2}, \quad (1.15)$$

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k} \left( x^k - \left( \frac{-1}{p} \right) (1-x)^k \right) \equiv 0 \pmod{p^2}. \quad (1.16)$$

Also,

$$\sum_{k=1}^{p-1} k \frac{\binom{3k}{k} \binom{2k}{k}}{27^k} \left( x^{k-1} + \left( \frac{p}{3} \right) (1-x)^{k-1} \right) \equiv 0 \pmod{p^2}, \quad (1.17)$$

$$\sum_{k=1}^{p-1} k \frac{\binom{4k}{2k} \binom{2k}{k}}{64^k} \left( x^{k-1} + \left( \frac{-2}{p} \right) (1-x)^{k-1} \right) \equiv 0 \pmod{p^2}, \quad (1.18)$$

$$\sum_{k=1}^{p-1} k \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k} \left( x^{k-1} + \left( \frac{-1}{p} \right) (1-x)^{k-1} \right) \equiv 0 \pmod{p^2}. \quad (1.19)$$

*Remark.* (1.17)-(1.19) can be easily deduced from (1.14)-(1.16) by taking derivations. Z. H. Sun [Su, Theorem 2.4] noted that for any prime  $p > 3$  we have

$$\sum_{k=0}^{\lfloor p/3 \rfloor} \frac{\binom{3k}{k} \binom{2k}{k}}{27^k} (x^k - (-1)^{\lfloor p/3 \rfloor} (1-x)^k) \equiv 0 \pmod{p}.$$

Taking  $x = 1/2$  in (1.14)-(1.19) we immediately get the following result.

**Corollary 1.1.** *Let  $p > 3$  be a prime. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} k \frac{\binom{3k}{k} \binom{2k}{k}}{54^k} &\equiv 0 \pmod{p^2} \quad \text{if } p \equiv 1 \pmod{3}, \\ \sum_{k=0}^{p-1} \frac{\binom{3k}{k} \binom{2k}{k}}{54^k} &\equiv 0 \pmod{p^2} \quad \text{if } p \equiv 2 \pmod{3}; \\ \sum_{k=0}^{p-1} k \frac{\binom{4k}{2k} \binom{2k}{k}}{128^k} &\equiv 0 \pmod{p^2} \quad \text{if } p \equiv 1, 3 \pmod{8}, \\ \sum_{k=0}^{p-1} \frac{\binom{4k}{2k} \binom{2k}{k}}{128^k} &\equiv 0 \pmod{p^2} \quad \text{if } p \equiv 5, 7 \pmod{8}; \end{aligned}$$

$$\sum_{k=0}^{p-1} \frac{k \binom{6k}{3k} \binom{3k}{k}}{864^k} \equiv 0 \pmod{p^2} \quad \text{if } p \equiv 1 \pmod{4},$$

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{864^k} \equiv 0 \pmod{p^2} \quad \text{if } p \equiv 3 \pmod{4}.$$

*Remark.* The first and the second congruences mod  $p$  were obtained by Z. H. Sun [Su]. The author [S4] and Z. H. Sun [Su] conjectured the first and the second congruences respectively. **Mathematica** yields that

$$\sum_{k=0}^{\infty} \frac{k \binom{2k}{k} \binom{3k}{k}}{54^k} = \frac{\sqrt{\pi}}{9\Gamma(\frac{4}{3})\Gamma(\frac{7}{6})}.$$

(1.14) and (1.17) in the case  $x = 9/8$ , and (1.15) and (1.18) in the cases  $x = 4/3, 8/9, 64/63$ , yield the following result.

**Corollary 1.2.** *Let  $p > 3$  be a prime. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{3k}{k} \binom{2k}{k}}{24^k} &\equiv \left(\frac{p}{3}\right) \sum_{k=0}^{p-1} \frac{\binom{3k}{k} \binom{2k}{k}}{216^k} \pmod{p^2}, \\ \sum_{k=0}^{p-1} \frac{k \binom{3k}{k} \binom{2k}{k}}{24^k} &\equiv -9 \left(\frac{p}{3}\right) \sum_{k=0}^{p-1} \frac{k \binom{3k}{k} \binom{2k}{k}}{216^k} \pmod{p^2}; \\ \sum_{k=0}^{p-1} \frac{\binom{4k}{2k} \binom{2k}{k}}{48^k} &\equiv \left(\frac{-2}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{4k}{2k} \binom{2k}{k}}{(-192)^k} \pmod{p^2}, \\ \sum_{k=0}^{p-1} \frac{k \binom{4k}{2k} \binom{2k}{k}}{48^k} &\equiv 4 \left(\frac{-2}{p}\right) \sum_{k=0}^{p-1} \frac{k \binom{3k}{k} \binom{2k}{k}}{(-192)^k} \pmod{p^2}; \\ \sum_{k=0}^{p-1} \frac{\binom{4k}{2k} \binom{2k}{k}}{72^k} &\equiv \left(\frac{-2}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{4k}{2k} \binom{2k}{k}}{576^k} \pmod{p^2}, \\ \sum_{k=0}^{p-1} \frac{k \binom{4k}{2k} \binom{2k}{k}}{72^k} &\equiv -8 \left(\frac{-2}{p}\right) \sum_{k=0}^{p-1} \frac{k \binom{3k}{k} \binom{2k}{k}}{576^k} \pmod{p^2}. \end{aligned}$$

If  $p \neq 7$  then

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{4k}{2k} \binom{2k}{k}}{63^k} &\equiv \left(\frac{-2}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{4k}{2k} \binom{2k}{k}}{(-4032)^k} \pmod{p^2}, \\ \sum_{k=0}^{p-1} \frac{k \binom{4k}{2k} \binom{2k}{k}}{63^k} &\equiv 64 \left(\frac{-2}{p}\right) \sum_{k=0}^{p-1} \frac{k \binom{3k}{k} \binom{2k}{k}}{(-4032)^k} \pmod{p^2}. \end{aligned}$$

*Remark.* In [S4, Conjecture 5.13] the author conjectured that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{24^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{(-216)^k} \equiv \begin{cases} \binom{(2(p-1)/3)}{(p-1)/3} \pmod{p^2} & \text{if } p \equiv 1 \pmod{6}, \\ 0 \pmod{p} & \text{if } p \equiv 5 \pmod{6}. \end{cases}$$

The author [S4] also made conjectures on  $\sum_{k=0}^{p-1} \binom{4k}{2k} \binom{2k}{k} / m^k$  modulo  $p^2$  or  $p$  with  $m = 48, 63, 72, 128$ ; the mod  $p$  case has been confirmed by Z. H. Sun recently.

For any prime  $p > 3$  and integer  $m \not\equiv 0 \pmod{p}$ , we have

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{3k}{k} C_k}{m^k} &\equiv p + \frac{m-27}{6} \sum_{k=0}^{p-1} \frac{k \binom{3k}{k} \binom{2k}{k}}{m^k} \pmod{p^2}, \\ \sum_{k=0}^{p-1} \frac{\binom{4k}{2k} C_k}{m^k} &\equiv p + \frac{m-64}{12} \sum_{k=0}^{p-1} \frac{k \binom{4k}{2k} \binom{2k}{k}}{m^k} \pmod{p^2}, \\ \sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{(k+1)m^k} &\equiv p + \frac{m-432}{60} \sum_{k=0}^{p-1} \frac{k \binom{6k}{3k} \binom{3k}{k}}{m^k} \pmod{p^2}, \end{aligned}$$

due to the identities

$$\begin{aligned} \sum_{k=0}^{n-1} \left( 6C_k + (27-m)k \binom{2k}{k} \right) \frac{\binom{3k}{k}}{m^k} &= \frac{n}{m^{n-1}} \binom{2n}{n} \binom{3n}{n}, \\ \sum_{k=0}^{n-1} \left( 12C_k + (64-m)k \binom{2k}{k} \right) \frac{\binom{4k}{2k}}{m^k} &= \frac{n}{m^{n-1}} \binom{4n}{2n} \binom{2n}{n}, \\ \sum_{k=0}^{n-1} \left( \frac{60}{k+1} + (432-m)k \right) \frac{\binom{6k}{3k} \binom{3k}{k}}{m^k} &= \frac{n}{m^{n-1}} \binom{6n}{3n} \binom{3n}{n}, \end{aligned}$$

which can be easily proved by induction on  $n$ . So, the following result follows from Corollary 1.1 and the second congruence in Corollary 1.2.

**Corollary 1.3.** *Let  $p > 3$  be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{3k}{k} C_k}{54^k} \equiv p \pmod{p^2} \quad \text{if } p \equiv 1 \pmod{3}, \quad (1.20)$$

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{2k} C_k}{128^k} \equiv p \pmod{p^2} \quad \text{if } p \equiv 1, 3 \pmod{8}, \quad (1.21)$$

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{(k+1)864^k} \equiv p \pmod{p^2} \quad \text{if } p \equiv 1 \pmod{4}. \quad (1.22)$$



We also have

$$\sum_{k=0}^{p-1} \frac{\binom{3k}{k} C_k}{24^k} \equiv p + \frac{1}{9} \left( \frac{p}{3} \right) \left( \sum_{k=0}^{p-1} \frac{\binom{3k}{k} C_k}{(-216)^k} - p \right) \pmod{p^2}. \quad (1.23)$$

*Remark.* Via **Mathematica** we find that

$$\sum_{k=0}^{\infty} \frac{\binom{3k}{k} C_k}{54^k} = \frac{3\sqrt{\pi}}{\Gamma(\frac{4}{3})\Gamma(\frac{1}{6})}, \quad \sum_{k=0}^{\infty} \frac{\binom{4k}{2k} C_k}{128^k} = \frac{4\sqrt{\pi}}{\Gamma(\frac{1}{8})\Gamma(\frac{11}{8})},$$

and

$$\sum_{k=0}^{\infty} \frac{\binom{6k}{3k} \binom{3k}{k}}{(k+1)864^k} = \frac{6\sqrt{\pi}}{\Gamma(\frac{1}{12})\Gamma(\frac{17}{12})}.$$

**Theorem 1.6.** *Let  $p > 3$  be a prime. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{k \binom{4k}{2k} \binom{2k}{k}}{72^k} &\equiv \frac{3}{2} \sum_{k=0}^{p-1} \frac{\binom{4k}{2k} C_k}{72^k} \\ &\equiv \begin{cases} \left( \frac{6}{p} \right) x \pmod{p} & \text{if } p = x^2 + y^2 \ (4 \mid x-1 \ \& \ 2 \mid y), \\ \frac{3}{4} \left( \frac{6}{p} \right) \binom{(p+1)/2}{(p+1)/4} \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \end{aligned}$$

## 2. PROOFS OF THEOREMS 1.1-1.3

*Proof of Theorem 1.1.* Set  $n = (p-1)/2$ . Then

$$\begin{aligned} a_p^{(d)}(\lambda) &\equiv \sum_{k=0}^{p-1} x^d (x(x-1)(x-\lambda))^n \\ &= \sum_{k=0}^{p-1} x^{n+d} \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} x^k \sum_{l=0}^n \binom{n}{l} (-\lambda)^l x^{n-l} \\ &= \sum_{k,l=0}^n \binom{n}{k} \binom{n}{l} (-1)^{n-k} (-\lambda)^l \sum_{x=1}^{p-1} x^{p-1+d+k-l} \\ &\equiv \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \sum_{\substack{0 \leq l \leq n \\ p-1 \mid l-(d+k)}} \binom{n}{l} (-\lambda)^l (p-1) \\ &\equiv - \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \binom{n}{d+k} (-\lambda)^{d+k} - \delta_{d,n} \binom{n}{0} (-\lambda)^0 \pmod{p}. \end{aligned}$$

Since

$$\binom{n}{k} \equiv \binom{-1/2}{k} = \frac{\binom{2k}{k}}{(-4)^k} \pmod{p} \quad \text{for all } k = 0, \dots, p-1,$$

we immediately obtain (1.2) from the above.  $\square$

*Proof of Theorem 1.2.* Let  $n = (p-1)/2$ . By **Mathematica**, for  $m = 0, \dots, n$  we have

$$\sum_{k=0}^n \frac{\binom{2k}{k}}{16^k} \left( \binom{2k}{k+m} - \binom{2k}{k+m+1} \right) = \frac{2n+1}{(2m+1)16^n} \binom{2n}{n} \binom{2n+1}{n-m}.$$

If  $0 \leq m < n$ , then for the right-hand side  $R_m$  of the last identity we have

$$\begin{aligned} R_m &= \frac{p^2}{(2m+1)((p-1)/2-m)4^{p-1}} \binom{p-1}{n} \binom{p-1}{n-m-1} \\ &\equiv 2p^2 \frac{(-1)^m}{(2m+1)^2} \pmod{p^3}. \end{aligned}$$

As  $d \leq n$ , we have

$$\begin{aligned} &\sum_{k=0}^n \frac{\binom{2k}{k}}{16^k} \left( \binom{2k}{k} - \binom{2k}{k+d} \right) \\ &= \sum_{0 \leq m < d} \sum_{k=0}^n \frac{\binom{2k}{k}}{16^k} \left( \binom{2k}{k+m} - \binom{2k}{k+m+1} \right) \\ &\equiv 2p^2 \sum_{0 \leq m < d} \frac{(-1)^m}{(2m+1)^2} \equiv \frac{p^2}{2} \sum_{0 \leq m < d} (-1)^m \left( m + \frac{1}{2} \right)^{p-3} \\ &\equiv \frac{p^2}{4} \sum_{0 \leq m < d} (-1)^m \left( E_{p-3} \left( m + \frac{1}{2} \right) + E_{p-3} \left( m + 1 + \frac{1}{2} \right) \right) \\ &= \frac{p^2}{4} \left( E_{p-3} \left( \frac{1}{2} \right) - (-1)^d E_{p-3} \left( d + \frac{1}{2} \right) \right) \pmod{p^3}. \end{aligned}$$

Note that

$$\sum_{k=0}^n \frac{\binom{2k}{k} \binom{2k}{k+n}}{16^k} = \frac{\binom{2n}{n}}{16^n} = \frac{\binom{p-1}{(p-1)/2}}{4^{p-1}} \equiv \left( \frac{-1}{p} \right) = (-1)^n \pmod{p^3}$$

by Morley's congruence ([M]), and  $E_{p-3}(n+1/2) = E_{p-3}(p/2) \equiv E_{p-3}(0) = 0 \pmod{p}$ . Therefore

$$\sum_{k=0}^n \frac{\binom{2k}{k}^2}{16^k} - (-1)^n \equiv \frac{p^2}{4} E_{p-3} \left( \frac{1}{2} \right) \equiv (-1)^n + p^2 E_{p-3} \pmod{p^3}$$

and hence (1.4) follows from the above.

For  $k = 0, 1, \dots$ , we have

$$\binom{2k+2d}{k+d} = \sum_{c=-d}^d \binom{2k}{k+c} \binom{2d}{d-c}$$

by the Chu-Vandermonde identity. Therefore

$$\begin{aligned} \sum_{k=0}^n \frac{\binom{2k}{k} \binom{2k+2d}{k+d}}{16^k} &= \sum_{c=-d}^d \binom{2d}{d-c} \sum_{k=0}^n \frac{\binom{2k}{k+c}^2}{16^k} \\ &\equiv \sum_{c=-d}^d \binom{2d}{d-c} \left( \frac{-1}{p} \right) = 2^{2d} \left( \frac{-1}{p} \right) \pmod{p^2}. \end{aligned}$$

So (1.3) is valid and we are done.  $\square$

*Proof of Theorem 1.3.* (i) For  $m \in \mathbb{Z} \setminus \{0\}$  and  $n \in \mathbb{N}$  we have the combinatorial identity

$$\sum_{k=0}^n \left( \frac{16-m}{4}k + \frac{1}{k+1} \right) \frac{\binom{2k}{k}^2}{m^k} = \frac{(2n+1)^2}{(n+1)m^n} \binom{2n}{n}^2$$

which can be easily proved by induction on  $n$ . Setting  $n = (p-1)/2$  we obtain from the identity that

$$\sum_{k=0}^n \frac{\binom{2k}{k} C_k}{m^k} \equiv \frac{m-16}{4} \sum_{k=0}^n \frac{k \binom{2k}{k}^2}{m^k} \pmod{p^2}$$

for any integer  $m \not\equiv 0 \pmod{p}$ .

As  $n = (p-1)/2$  is odd, by a result of Z. H. Sun [Su],

$$\sum_{k=0}^n \frac{\binom{2k}{k}^2}{16^k} (x^k + (1-x)^k) = p^2 f(x)$$

for some polynomial  $f(x)$  of degree at most  $(p-1)/2$  with rational  $p$ -adic integer coefficients. In particular,  $\sum_{k=0}^n \binom{2k}{k}^2 / 8^k \equiv -\sum_{k=0}^n \binom{2k}{k}^2 / (-16)^k \pmod{p^2}$ . By integration,

$$\sum_{k=0}^n \frac{\binom{2k}{k}}{(k+1)16^k} x^{k+1} - \sum_{k=0}^n \frac{\binom{2k}{k}^2}{(k+1)16^k} ((1-x)^{k+1} - 1) = p^2 \int_0^x f(t) dt.$$

Putting  $x = -1$  we obtain

$$-\sum_{k=0}^n \frac{\binom{2k}{k} C_k}{(-16)^k} - \sum_{k=0}^n \frac{\binom{2k}{k} C_k}{16^k} (2^{k+1} - 1) \equiv 0 \pmod{p^2}.$$

Since

$$\sum_{k=0}^n \frac{\binom{2k}{k} C_k}{16^k} = \frac{(2n+1)^2}{16(n+1)} \binom{2n}{n}^2 \equiv 0 \pmod{p^2},$$

as observed by van Hamme [vH], we have

$$\sum_{k=0}^n \frac{\binom{2k}{k} C_k}{(-16)^k} \equiv -2 \sum_{k=0}^n \frac{\binom{2k}{k} C_k}{8^k} \pmod{p^2}.$$

Clearly,

$$\sum_{k=0}^n \frac{\binom{2k}{k}^2}{(-16)^k} = \sum_{k=0}^n (-1)^k \binom{-1/2}{k}^2 \equiv \sum_{k=0}^n (-1)^k \binom{n}{k}^2 = 0 \pmod{p}.$$

(Note that  $(-1)^{n-k} = -(-1)^k$ .) Thus

$$\begin{aligned} & \sum_{h=0}^{p-1} \frac{2h+1}{(-16)^h} \sum_{k=0}^h \binom{2k}{k}^2 \binom{2(h-k)}{h-k}^2 \\ & \equiv \sum_{k=0}^n \frac{\binom{2k}{k}^2}{(-16)^k} \sum_{j=0}^n \frac{(2(k+j)+1) \binom{2j}{j}^2}{(-16)^j} \equiv 4 \sum_{k=0}^n \frac{\binom{2k}{k}^2}{(-16)^k} \sum_{j=0}^n \frac{j \binom{2j}{j}^2}{(-16)^j} \pmod{p^2}. \end{aligned}$$

By [S5, Lemma 3.1],

$$\sum_{h=0}^{p-1} \frac{2h+1}{(-16)^h} \sum_{k=0}^h \binom{2k}{k}^2 \binom{2(h-k)}{h-k}^2 \equiv p \left( \frac{-1}{p} \right) = -p \pmod{p^2}.$$

Therefore

$$\frac{1}{p} \sum_{k=0}^n \frac{\binom{2k}{k}^2}{(-16)^k} \times \sum_{k=0}^n \frac{k \binom{2k}{k}^2}{(-16)^k} \equiv -\frac{1}{4} \pmod{p}.$$

In view of the above, both (1.5) and (1.6) hold if

$$\sum_{k=0}^n \frac{\binom{2k}{k} C_k}{8^k} \equiv \frac{(-1)^{(p+1)/4}}{2} \binom{(p+1)/2}{(p+1)/4} \pmod{p}. \quad (2.1)$$

For  $d = 0, 1$  clearly

$$a_p^{(d)}(2) = \sum_{x=1}^p x^d \left( \frac{x(x-1)(x-2)}{p} \right) = \sum_{r=0}^{p-1} (r+1)^d \left( \frac{r(r^2-1)}{p} \right)$$

and

$$\begin{aligned} a_p^{(d)}(-1) &= \sum_{r=0}^{p-1} r^d \left( \frac{r(r^2-1)}{p} \right) \\ &\equiv \sum_{r=0}^{p-1} r^{d+n} (r^2-1)^n = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \sum_{r=1}^{p-1} r^{n+d+2k} \\ &\equiv - \sum_{\substack{0 \leq k \leq n \\ p-1 \mid n+d+2k}} \binom{n}{k} (-1)^{n-k} \\ &\equiv \begin{cases} 0 \pmod{p} & \text{if } d = 0, \\ (-1)^{(p-3)/4} \binom{n}{(n-1)/2} \pmod{p} & \text{if } d = 1. \end{cases} \end{aligned}$$

Thus we have

$$a_p^{(0)}(2) = a_p^{(0)}(-1) \equiv 0 \pmod{p}$$

and

$$a_p^{(1)}(2) = a_p^{(0)}(-1) + a_p^{(1)}(-1) \equiv (-1)^{(p-3)/4} \binom{n}{(n-1)/2} \pmod{p}.$$

Applying Theorem 1.1 with  $\lambda = 2$  and  $d = 0, 1$ , and noting that

$$\frac{1}{2} \binom{2k+2}{k+1} = \binom{2k+1}{k+1} = \binom{2k}{k} + \binom{2k}{k+1} = 2 \binom{2k}{k} - C_k \quad (k = 0, 1, \dots),$$

we get

$$\sum_{k=0}^n \frac{\binom{2k}{k} C_k}{8^k} + \delta_{p,3} \equiv 2a_p^{(0)}(2) - a_p^{(1)}(2) \pmod{p}.$$

So (2.1) follows.

(ii) Now we prove (1.7) for all  $d \in \{0, \dots, n\}$  with  $d \equiv n+1 \pmod{2}$ , where  $n = (p-1)/2$ . (1.7) is valid for  $d = n-1$  since

$$\sum_{k=0}^n \frac{\binom{2k}{k} \binom{2k}{k+n-1}}{8^k} = \frac{\binom{2(n-1)}{n-1}}{8^{n-1}} + \frac{2n \binom{2n}{n}}{8^n} = \frac{2n+1}{2 \times 8^{n-1}} \binom{2n-2}{n-1} \equiv 0 \pmod{p}.$$

Define

$$f(d) := \sum_{k=0}^n \binom{n+k}{2k} \binom{2k}{k+d} (-2)^k \quad \text{for } d = 0, 1, \dots$$

Since

$$\binom{n+k}{2k} \equiv \frac{\binom{2k}{k}}{(-16)^k} \pmod{p^2} \quad \text{for } k = 0, \dots, n$$

(see, e.g., [Su, Lemma 2.2]), we have

$$f(d) \equiv \sum_{k=0}^n \frac{\binom{2k}{k} \binom{2k}{k+d}}{8^k} \pmod{p^2}$$

for all  $d = 0, \dots, n$ . By the Zeilberger algorithm,

$$\begin{aligned} & (n-d-1)(n+d+2)(2d+1)f(d+2) \\ &= (2n+1)^2(d+1)f(d+1) - (n-d)(n+d+1)(2d+3)f(d). \end{aligned}$$

Note that  $2n+1 = p$ . So, if  $0 \leq d \leq n-2$ , then

$$f(d) \equiv -\frac{(n-d-1)(n+d+2)(2d+1)}{(n-d)(n+d+1)(2d+3)} f(d+2) \pmod{p^2}$$

and hence

$$f(d+2) \equiv 0 \pmod{p} \implies f(d) \equiv 0 \pmod{p}.$$

Now it is clear that (1.7) holds for all  $d \in \{0, \dots, n\}$  with  $d \equiv n+1 \pmod{2}$ .

### 3. PROOF OF THEOREM 1.4

*Proof of (1.11).* Observe that

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{3k}{k} \binom{2k}{k}}{27^k} a_k^* &= \sum_{k=0}^{p-1} \frac{\binom{3k}{k} \binom{2k}{k}}{27^k} \sum_{m=0}^k \binom{k}{m} (-1)^m \\ &= \sum_{m=0}^{p-1} (-1)^m a_m \sum_{k=m}^{p-1} \frac{\binom{3k}{k} \binom{2k}{k}}{27^k} \binom{k}{m}. \end{aligned}$$

So it suffices to show that

$$\sum_{k=m}^{p-1} \frac{\binom{3k}{k} \binom{2k}{k}}{27^k} \binom{k}{m} \equiv \left(\frac{p}{3}\right) \frac{\binom{3m}{m} \binom{2m}{m}}{(-27)^m} \pmod{p^2}$$

for all  $m = 0, 1, \dots, p-1$ .

For  $0 \leq m < n$  define

$$f_n(m) = \sum_{k=m}^{n-1} \frac{\binom{3k}{k} \binom{2k}{k}}{27^k} \binom{k}{m}.$$

By Zeilberger's algorithm (see, e.g., [PWZ] for this method) via the software **Mathematica 7** (version 7),

$$\begin{aligned} & 9(m+1)^2 f_n(m+1) + (3m+1)(3m+2) f_n(m) \\ &= \frac{(3n-1)(3n-2)}{27^{n-1}} \binom{n-1}{m} \binom{2n-2}{n-1} \binom{3n-3}{n-1}. \end{aligned}$$

Applying this with  $n = p > m+1 \geq 1$  and noting that

$$\binom{2p-2}{p-1} = \frac{p}{2p-1} \binom{2p-1}{p-1} \equiv -p \pmod{p^2}$$

and

$$\binom{3p-3}{p-1} = \frac{p}{3p-2} \binom{3p-2}{p-2} \equiv -\frac{p}{2} \pmod{p^2},$$

we get

$$\begin{aligned} & 9(m+1)^2 f_p(m+1) + (3m+1)(3m+2) f_p(m) \\ & \equiv \frac{(3p-1)(3p-2)}{27^{p-1}} \binom{p-1}{m} \frac{p^2}{2} \equiv (-1)^m p^2 \pmod{p^3} \end{aligned}$$

and hence

$$\begin{aligned} & f_p(m+1) - \left(\frac{p}{3}\right) \frac{\binom{3m+3}{m+1} \binom{2m+2}{m+1}}{(-27)^{m+1}} \\ & + \frac{(3m+1)(3m+2)}{9(m+1)^2} \left( f_p(m) - \left(\frac{p}{3}\right) \frac{\binom{3m}{m} \binom{2m}{m}}{(-27)^m} \right) \\ & = f_p(m+1) + \frac{(3m+1)(3m+2)}{9(m+1)^2} f_p(m) \equiv p^2 \frac{(-1)^m}{9(m+1)^2} \pmod{p^3}. \end{aligned}$$

Thus

$$\begin{aligned} & f_p(m) \equiv \left(\frac{p}{3}\right) \frac{\binom{3m}{m} \binom{2m}{m}}{(-27)^m} \pmod{p^2} \\ \implies & f_p(m+1) \equiv \left(\frac{p}{3}\right) \frac{\binom{3(m+1)}{m+1} \binom{2(m+1)}{m+1}}{(-27)^{m+1}} \pmod{p^2}. \end{aligned} \tag{3.1}$$

Since

$$f_p(0) = \sum_{k=0}^{p-1} \frac{\binom{3k}{k} \binom{2k}{k}}{27^k} \equiv \left(\frac{p}{3}\right) \frac{\binom{3 \times 0}{0} \binom{2 \times 0}{0}}{(-27)^0} \pmod{p^2}$$

by (1.8), from the above we obtain that

$$f_p(m) \equiv \left(\frac{p}{3}\right) \frac{\binom{3m}{m} \binom{2m}{m}}{(-27)^m} \pmod{p^2} \quad \text{for all } m = 0, 1, \dots, p-1.$$

This concludes the proof.  $\square$

*Proof of (1.12).* Similar to the proof of (1.6), we only need to show that

$$\sum_{k=m}^{p-1} \frac{\binom{4k}{2k} \binom{2k}{k}}{64^k} \binom{k}{m} \equiv \left(\frac{-2}{p}\right) \frac{\binom{4m}{2m} \binom{2m}{m}}{(-64)^m} \pmod{p^2}$$

for all  $m = 0, 1, \dots, p-1$ . Since the last congruence holds for  $m = 0$  by (1.3), it suffices to prove that for any fixed  $0 \leq m < p-1$  we have

$$\begin{aligned} g_p(m) &\equiv \left(\frac{-2}{p}\right) \frac{\binom{4m}{2m} \binom{2m}{m}}{(-64)^m} \pmod{p^2} \\ \implies g_p(m+1) &\equiv \left(\frac{-2}{p}\right) \frac{\binom{4(m+1)}{2(m+1)} \binom{2(m+1)}{m+1}}{(-64)^{m+1}} \pmod{p^2}. \end{aligned} \tag{3.2}$$

where

$$g_n(m) := \sum_{k=m}^{n-1} \frac{\binom{4k}{2k} \binom{2k}{k}}{64^k} \binom{k}{m}$$

with  $n > m$ . By the Zeilberger algorithm, we find that

$$\begin{aligned} &16(m+1)^2 g_n(m+1) + (4m+1)(4m+3) g_n(m) \\ &= \frac{(4n-1)(4n-3)}{64^{n-1}} \binom{n-1}{m} \binom{2n-2}{n-1} \binom{4n-4}{2n-2}. \end{aligned}$$

Recall the congruence  $\binom{2p-2}{p-1} \equiv -p \pmod{p^2}$  and note that

$$\binom{4p-4}{2p-2} = \frac{2p(2p-1)}{(4p-2)(4p-3)} \binom{4p-2}{2p-2} \equiv p \pmod{p^2}.$$

So we have

$$16(m+1)^2 g_p(m+1) + (4m+1)(4m+3) g_p(m) \equiv 3(-1)^m (-p^2) \pmod{p^3}.$$



This implies (3.2) since

$$-\frac{(4m+1)(4m+3)}{16(m+1)^2} \cdot \frac{\binom{4m}{2m}\binom{2m}{m}}{(-64)^m} = \frac{\binom{4(m+1)}{2(m+1)}\binom{2(m+1)}{m+1}}{(-64)^{m+1}}.$$

We are done.  $\square$

*Proof of (1.13).* For  $0 \leq m < n$  define

$$h_n(m) := \sum_{k=m}^{n-1} \frac{\binom{6k}{3k}\binom{3k}{k}}{432^k} \binom{k}{m}.$$

By the Zeilberger algorithm we have

$$\begin{aligned} & 36(m+1)^2 h_n(m+1) + (6m+1)(6m+5) h_n(m) \\ &= \frac{(6n-1)(6n-5)}{432^{n-1}} \binom{n-1}{m} \binom{3n-3}{n-1} \binom{6n-6}{3n-3}. \end{aligned}$$

Recall the congruence  $\binom{3p-3}{p-1} \equiv -p/2 \pmod{p^2}$  and note that

$$\begin{aligned} \binom{6p-6}{3p-3} &= \frac{3p(3p-1)(3p-2)}{(6p-3)(6p-4)(6p-5)} \binom{6p-3}{3p-3} \\ &\equiv -\frac{p}{10} \binom{5p+(p-3)}{2p+(p-3)} \equiv -\frac{p}{10} \binom{5}{2} = -p \pmod{p^2} \end{aligned}$$

if  $p > 5$ . Whether  $p = 5$  or not, we always have

$$36(m+1)^2 h_p(m+1) + (6m+1)(6m+5) h_p(m) \equiv 0 \pmod{p^2}.$$

For  $0 \leq m < p-1$ , since

$$-\frac{(6m+1)(6m+5)}{36(m+1)^2} \cdot \frac{\binom{6m}{3m}\binom{3m}{m}}{(-432)^m} = \frac{\binom{6(m+1)}{3(m+1)}\binom{3(m+1)}{m+1}}{(-432)^{m+1}},$$

by the above we have

$$\begin{aligned} h_p(m) &\equiv \left(\frac{-1}{p}\right) \frac{\binom{6m}{3m}\binom{3m}{m}}{(-432)^m} \pmod{p^2} \\ \implies h_p(m+1) &\equiv \left(\frac{-1}{p}\right) \frac{\binom{6(m+1)}{3(m+1)}\binom{3(m+1)}{m+1}}{(-432)^{m+1}} \pmod{p^2}. \end{aligned} \tag{3.3}$$

This together with (1.4) yields that

$$h_p(m) \equiv \left(\frac{-1}{p}\right) \frac{\binom{6m}{3m}\binom{3m}{m}}{(-432)^m} \pmod{p^2}$$

for all  $m = 0, \dots, m-1$ . It follows that

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k} \sum_{m=0}^k \binom{k}{m} (-1)^m a_m \\ &= \sum_{m=0}^{p-1} (-1)^m a_m h_p(m) \equiv \left( \frac{-1}{p} \right) \sum_{m=0}^{p-1} a_m \frac{\binom{6m}{3m} \binom{3m}{m}}{(-432)^m} \pmod{p^2}. \end{aligned}$$

This proves (1.13).  $\square$

#### 4. PROOF OF THEOREM 1.5

Recall that

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{2k} C_k}{72^k} \equiv p + \frac{72-64}{12} \sum_{k=0}^{p-1} \frac{k \binom{4k}{2k} \binom{2k}{k}}{72^k} \pmod{p^2}$$

and hence

$$\sum_{k=0}^{p-1} \frac{k \binom{4k}{2k} \binom{2k}{k}}{72^k} \equiv \frac{3}{2} \sum_{k=0}^{p-1} \frac{\binom{4k}{2k} C_k}{72^k} \pmod{p}.$$

Below we determine  $\sum_{k=0}^n k \binom{4k}{2k} \binom{2k}{k} / 72^k \pmod{p}$ , where  $n = (p-1)/2$ . (Note that  $p \mid \binom{2k}{k}$  for  $k = n+1, \dots, p-1$ .)

The Legendre polynomial of degree  $n$  is given by

$$P_n(x) := \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \left( \frac{x-1}{2} \right)^k = \sum_{k=0}^n \binom{n+k}{2k} \binom{2k}{k} \left( \frac{x-1}{2} \right)^k.$$

It is known that

$$\sum_{k=0}^n \binom{n}{2k} \binom{2k}{k} x^k = (\sqrt{1-4x})^n P_n \left( \frac{1}{\sqrt{1-4x}} \right).$$

(Note that the left-hand side is the coefficient of  $t^n$  in  $(t^2+t+x)^n$ .) Taking derivations of both sides of the last equality, we get

$$\begin{aligned} & \sum_{k=0}^n \binom{n}{2k} \binom{2k}{k} k x^{k-1} \\ &= -2n(1-4x)^{n/2-1} \sum_{k=0}^n \binom{n+k}{2k} \binom{2k}{k} \left( \frac{(1-4x)^{-1/2} - 1}{2} \right)^k \\ & \quad + (1-4x)^{(n-3)/2} \sum_{k=0}^n \binom{n+k}{2k} \binom{2k}{k} k \left( \frac{(1-4x)^{-1/2} - 1}{2} \right)^{k-1} \end{aligned}$$

Since

$$\binom{n}{2k} \equiv \binom{-1/2}{2k} = \frac{\binom{4k}{2k}}{(-4)^{2k}} \pmod{p} \text{ and } \binom{n+k}{2k} \equiv \frac{\binom{2k}{k}}{(-16)^k} \pmod{p^2}$$

for all  $k = 0, \dots, n$ , by putting  $x = 2/9$  in the last equality we obtain

$$\frac{1}{2} \sum_{k=0}^n \frac{k \binom{4k}{2k} \binom{2k}{k}}{72^k} \equiv \frac{1}{3^n} \sum_{k=0}^n \frac{\binom{2k}{k}^2}{(-16)^k} + \frac{3}{3^n} \sum_{k=0}^n \frac{k \binom{2k}{k}^2}{(-16)^k} \pmod{p}$$

and hence

$$\left(\frac{3}{p}\right) \sum_{k=0}^n \frac{k \binom{4k}{2k} \binom{2k}{k}}{72^k} \equiv 2 \sum_{k=0}^n \frac{\binom{2k}{k}^2}{16^k} + 6 \sum_{k=0}^n \frac{k \binom{2k}{k}^2}{(-16)^k} \pmod{p}.$$

Thus, with the help of Theorem 1.5 and the related known results for the case  $p \equiv 1 \pmod{4}$ , we finally obtain the desired result.

The proof of Theorem 1.5 is now complete.

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